

II Homology Theory

There are many types of (ordinary) homology theories

singular, simplicial, cubical, cellular, ...

these all give the same results on CW-complexes

we will discuss the "most general" one singular homology

and then derive an easily computable one cellular homology

There are also generalized homologies, like bordism theory, K-theory ...

these are different from ordinary homology

we might discuss them briefly

Underlying these theories is homological algebra which is a purely

algebraic theory of "chain complexes"

such objects show up in many contexts and give even more "homology theories", like Floer-homology

such theories are not really about algebraic topology

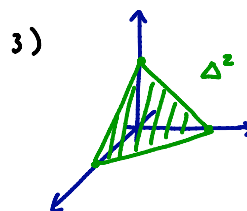
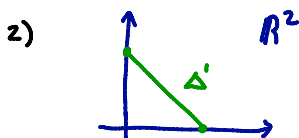
A Singular Homology

the standard p -simplex is

$$\Delta^p = \left\{ \sum_{i=0}^p t_i e_i \in \mathbb{R}^{p+1} : \sum_{i=0}^p t_i = 1, t_i \geq 0 \right\}$$

where $e_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$, $e_1 = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$, ..., $e_p = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$ are the standard basis vectors for \mathbb{R}^{p+1}

examples:



given $v_0, \dots, v_n \in \mathbb{R}^n$ denote by $[v_0, \dots, v_p]$ the map

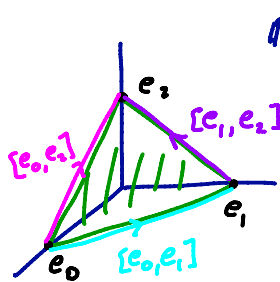
$$\begin{aligned} \Delta^p &\rightarrow \mathbb{R}^n \\ (t_0, \dots, t_p) &\mapsto \sum t_i v_i \end{aligned}$$

example:

- 1) $[e_0, \dots, e_p]: \Delta^p \rightarrow \mathbb{R}^{p+1}$ parameterizes Δ^p
- 2) $[e_0, \dots, \hat{e}_i, \dots, e_p]: \Delta^{p-1} \rightarrow \Delta^p$ parameterizes 1th face of Δ^p

means leave out i th element
 i.e. $(t_0, \dots, t_{p-1}) \in \Delta^{p-1} \mapsto \sum_{j \neq i} t_j e_j + \sum_{j \geq i} t_j e_{j+1}$

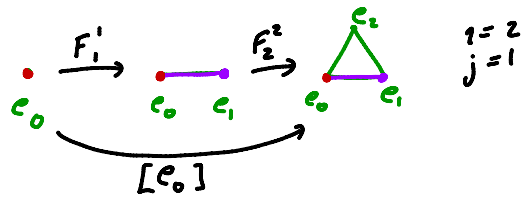
eg. $p=2$



we call this the 1th face map and also denote it $F_i^p = [e_0, \dots, \hat{e}_i, \dots, e_p]$

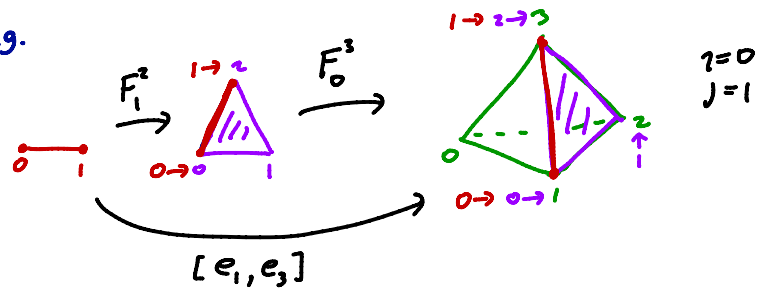
note: $i > j$ then $F_i^p \circ F_j^{p-1} = [e_0, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_p]$

eg.



$i \leq j$ then $F_i^p \circ F_j^{p-1} = [e_0, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_p]$

eg.



a singular p -simplex in a space X is a continuous map

$$\sigma: \Delta^p \rightarrow X$$

the singular group of p -chains in X is

$C_p(X) =$ free abelian group generated by singular p -simplices

that is an element of $C_p(X)$ is a finite formal sum

$$\sum_{i=1}^k n_i \sigma_i \quad \text{where } n_i \in \mathbb{Z} \text{ and } \sigma_i: \Delta^p \rightarrow X \quad \forall i$$

exercise: There is an obvious way to add two elements
Show this makes $C_p(X)$ an abelian group.

we call elements of $C_p(X)$ (singular) p-chains

given a singular p-simplex σ we say the ith face of σ is

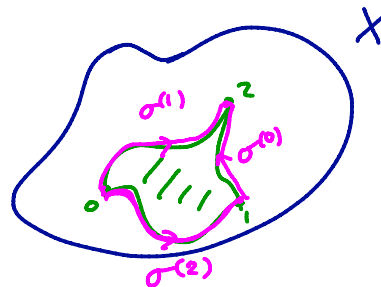
$$\sigma^{(i)} = \sigma \circ F_i^p$$

and the boundary of σ is

$$\partial \sigma = \sum_{i=0}^p (-1)^i \sigma^{(i)}$$

note: $\partial \sigma$ is a $(p-1)$ -chain!

example:



$$\partial \sigma = \sigma^{(0)} - \sigma^{(1)} + \sigma^{(2)}$$

now define the boundary map

$$\begin{aligned} \partial_p: C_p(X) &\rightarrow C_{p-1}(X) \\ \sum_{i=1}^k n_i \sigma_i &\mapsto \sum_{i=1}^k n_i (\partial \sigma_i) \end{aligned}$$

lemma 1:

With notation as above

$$\partial_{p-1} \circ \partial_p = 0$$

subscript p is usually omitted so
lemma then stated $\partial^2 = 0$

Proof:

$$\begin{aligned} \partial_{p-1} \circ \partial_p \sigma &= \partial_{p-1} \left(\sum_{i=0}^p (-1)^i \sigma \circ F_i^p \right) = \sum_{i=0}^p (-1)^i \sum_{j=0}^{p-1} (-1)^j \sigma \circ F_i^p \circ F_j^{p-1} \\ &= \sum_{0 \leq j < i \leq p} (-1)^{i+j} \sigma \circ F_i^p \circ F_j^{p-1} + \sum_{0 \leq i < j \leq p-1} (-1)^{i+j} \sigma \circ F_i^p \circ F_j^{p-1} \end{aligned}$$

$$\begin{aligned}
&= \sum_{0 \leq j < i \leq p} (-1)^{i+j} \sigma_0[e_0 \dots \hat{e}_i \dots \hat{e}_j \dots e_p] + \sum_{0 \leq i < j \leq p-1} (-1)^{i+j} \sigma_0[e_0, \dots, \hat{e}_i \dots \hat{e}_{j+1} \dots e_n] \\
&\quad \downarrow k=j+1 \\
&= \sum_{0 \leq j < i \leq p} (-1)^{i+j} \sigma_0[e_0 \dots \hat{e}_i \dots \hat{e}_j \dots e_p] + \sum_{0 \leq i < k \leq p} (-1)^{i+k-1} \sigma_0[e_0 \dots \hat{e}_i \dots \hat{e}_k \dots e_n] \\
&\quad \text{note relable back to } j \\
&= \sum_{j < i} (-1)^{i+j} \sigma_0[e_0 \dots \hat{e}_i \dots \hat{e}_j \dots e_p] - \sum_{i < j} (-1)^{i+j} \sigma_0[e_0 \dots \hat{e}_i \dots \hat{e}_j \dots e_n] \\
&= 0 \quad \square
\end{aligned}$$

note: lemma \Rightarrow image $\partial_p \subset$ kernel ∂_{p-1}

we define the p^{th} homology group of X to be

$$H_p(X) = \ker \partial_p / \text{im } \partial_{p+1}$$

an element of $\ker \partial_p$ is called a (singular) p -cycle

an element of $\text{im } \partial_{p+1}$ is called a (singular) p -boundary

we say two chains $c_1, c_2 \in C_p(X)$ are homologous if \exists some $d \in C_{p+1}(X)$ st $c_1 - c_2 = \partial d$, we write $c_1 \sim c_2$

this is an equivalence relation if c is a p -cycle

let $[c]$ be its equivalence class

$$H_p(X) = \{[c] \mid c \text{ a } p\text{-cycle}\}$$

Remark: There are no singular p -simplices for $p < 0$ so $C_p(X) = \{0\}$

$$\therefore H_p(X) = 0 \quad \forall p < 0$$

lemma 2:

let $X =$ one point space

then

$$H_p(X) = \begin{cases} \mathbb{Z} & p=0 \\ 0 & p \neq 0 \end{cases}$$

Proof: for each $p \geq 0 \exists!$ map $\sigma_p: \Delta^p \rightarrow X$

$$\therefore C_p(X) = \mathbb{Z} \text{ generated by } \sigma_p$$

now

$$\partial_p \sigma_p = \sum_{i=0}^p (-1)^i \sigma_p^{(i)} = \sum_{i=0}^p (-1)^i \sigma_{p-1}$$

$$\text{so } \partial_p \sigma_p = \begin{cases} \sigma_{p-1} & p \text{ even } \quad p > 0 \\ 0 & p \text{ odd or } p = 0 \end{cases}$$

$$\therefore p \text{ odd } \quad H_p(X) = \frac{\ker \partial_p}{\text{im } \partial_{p+1}} = \frac{C_p(X)}{C_p(X)} = \{0\}$$

$p \text{ even, } p > 0$

$$H_p(X) = \frac{\ker \partial_p}{\text{im } \partial_{p+1}} = \frac{\{0\}}{\{0\}} = \{0\}$$

$p = 0$

$$H_0(X) = \frac{\ker \partial_0}{\text{im } \partial_1} = \frac{C_0(X)}{\{0\}} = \frac{\mathbb{Z}}{\{0\}} \cong \mathbb{Z} \quad \square$$

Thm 3:

$H_0(X)$ = free abelian group generated by path components of X
 $\cong \bigoplus_n \mathbb{Z}$ where X has n path components

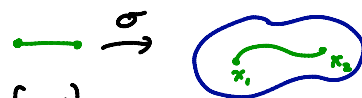
Proof: an element $c \in C_0(X)$ is $c = \sum_{i=1}^k n_i x_i$ for points $x_i \in X, n_i \in \mathbb{Z}$

define $\varepsilon: C_0(X) \rightarrow \mathbb{Z}$

$$\sum n_i x_i \mapsto \sum n_i$$

It is easy to check ε is a homomorphism (do it!)

If σ is a singular 1-simplex then



$$\partial \sigma = x_1 - x_2 \quad (\text{end points of } \sigma)$$

$$\text{so } \varepsilon(\partial \sigma) = 0$$

if $d \in C_1(X)$ then $d = \sum_{i=1}^k n_i \sigma_i$ so $\varepsilon(\partial d) = 0$

i.e. $\text{im } \partial \subset \ker \varepsilon$

so ε induces a homomorphism

$$\varepsilon_*: H_0(X) \rightarrow \mathbb{Z} \quad \text{called an augmentation (so is } \varepsilon) \\ [c] \mapsto \varepsilon(c)$$

Claim: if X is path connected then ε_* is an isomorphism

Pf: clearly ε_* is surjective ($\varepsilon_*([x]) = 1$)

fix $x_0 \in X$, then for any $x \in X$ we can take $\lambda_x: [0,1] \rightarrow X$ st. $\lambda_x(0) = x_0$
 $\lambda_x(1) = x$

$$\text{so } \partial \lambda_x = x - x_0$$

now given any $c = \sum n_i x_i$ such that $\varepsilon(c) = 0$

let λ_{x_i} be path for x_i

$$\text{note } c - \partial \sum n_i \lambda_{x_i} = \sum n_i x_i - \sum n_i (x_i - x_0) = \sum n_i x_0 = (\sum n_i) x_0 = 0$$

$$\text{so } c = \partial \sum n_i \lambda_{x_i} \text{ and } [c] = 0 \text{ in } H_0(X)$$

exercise: If the path components of X are $X_\alpha, \alpha \in A$

$$\text{then } C_p(X) = \bigoplus_{\alpha \in A} C_p(X_\alpha)$$

$$\text{and } H_p(X) = \bigoplus_{\alpha \in A} H_p(X_\alpha)$$

note lemma now follows

Remark: if we set $\tilde{\partial}_0 = \varepsilon$ and $\tilde{\partial}_i = \partial_i$ for $i \geq 1$ then the proof

$$\text{shows } \tilde{\partial}_i \circ \tilde{\partial}_{i-1} = 0 \quad \forall i \geq 0$$

$$\text{so we can define } \tilde{H}_p(X) = \frac{\ker \tilde{\partial}_p}{\text{im } \tilde{\partial}_{p+1}}$$

$$\text{Clearly } H_p(X) = \tilde{H}_p(X) \quad \forall p \geq 1 \text{ and}$$

$$H_0(X) = \tilde{H}_0(X) \oplus \mathbb{Z}$$

$\tilde{H}_p(X)$ is called the reduced homology of X .

notice that if $\gamma: [0,1] \rightarrow X$ is a loop based at x_0

then γ is also a singular 1-simplex and

$$\partial \gamma = 0 \quad \text{so } [\gamma] \in H_1(X)$$

this gives a map

$$\phi: \pi_1(X, x_0) \rightarrow H_1(X) \quad \text{called the Hurewicz map}$$

(we check it's well-defined below)

Thm 4:

If X path connected, then the Hurewicz map induces an isomorphism

$$\phi_*: (\pi_1(X, x_0))^{ab} \rightarrow H_1(X)$$

where $(\pi_1(X, x_0))^{ab}$ is the abelianization of $\pi_1(X, x_0)$

the abelianization G^{ab} of a group G is the largest abelian

quotient of G .

that is if A is any abelian group and $f: G \rightarrow A$ a homomorphism, then $\exists \tilde{f}: G^{ab} \rightarrow A$ s.t.

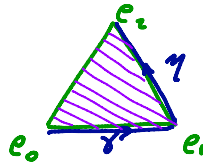
$$G \begin{array}{c} \xrightarrow{f} A \\ \searrow \downarrow \tilde{f} \\ G^{ab} \end{array}$$

exercise: $G^{ab} = G/[G, G]$ where $[G, G]$ is the smallest normal subgroup of G containing $\{[g, h] : g, h \in G\}$

Proof: we will denote equiv. classes in $\pi_1(X, x_0)$ by $[\gamma]$ and equiv. classes in $H_1(X)$ by $[\gamma]$

note: 1) If γ, η paths in X with $\gamma(1) = \eta(0)$, then $\gamma * \eta - \gamma - \eta$ is a boundary

indeed, define $\sigma: \Delta^2 \rightarrow X$ by



constant on pink lines
so on $[e_0, e_2]$ have $\gamma * \eta$

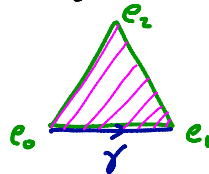
$$\text{now } \partial\sigma = \gamma - \gamma * \eta + \eta$$

2) If γ a path in X , then $\gamma + \bar{\gamma}$ is a boundary (and constant path a boundary)

indeed, if $\sigma: \Delta^2 \rightarrow X$ a constant map,

$$\text{then } \partial\sigma = \underbrace{\sigma^{(0)} - \sigma^{(1)} + \sigma^{(2)}}_{\text{each a constant path } c} = c$$

now given γ let $\sigma': \Delta^2 \rightarrow X$ be



constant on pink lines

so σ' on $[e_1, e_2]$ is $\bar{\gamma}$

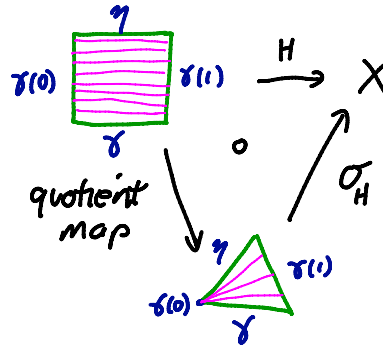
on $[e_0, e_2]$ is some constant path c

let σ be singular 2-simplex with $\partial\sigma = c$

$$\text{so } \partial(\sigma' - \sigma) = \gamma + \bar{\gamma}$$

3) if γ and η are homotopic rel end points then $\gamma - \eta$ is a boundary

indeed, let $H: [0,1] \times [0,1] \rightarrow X$ be the homotopy
 then H induces a singular 2-simplex σ_H



now $\partial\sigma = \gamma - \eta + \text{constant path}$

since a constant path is a boundary so is $\gamma - \eta$

now: 3) $\Rightarrow \phi$ is well defined

1) $\Rightarrow \phi$ a homomorphism:

$$\phi([\gamma] + [\eta]) = \phi([\gamma * \eta]) = [\gamma * \eta] \stackrel{(1)}{=} [\gamma] + [\eta] = \phi([\gamma]) + \phi([\eta])$$

since $H_1(X)$ abelian we get

$$\phi_x: (\pi_1(X, x_0))^{ab} \rightarrow H_1(X)$$

we construct an inverse for ϕ_x

for each point $x \in X$ let γ_x be a path x_0 to x

given a singular 1-simplex σ let

$$\hat{\sigma} = \gamma_{\sigma(0)} * \sigma * \bar{\gamma}_{\sigma(1)}$$

this is a loop in X based at x_0

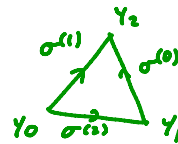
now define $\Psi(\sigma) = [\hat{\sigma}]$ since $(\pi_1(X, x_0))^{ab}$ is abelian and

$C_1(X)$ a free abelian group this defines

$$\Psi: C_1(X) \rightarrow (\pi_1(X, x_0))^{ab}$$

note $\Psi \circ \phi_x([\gamma]) = [e_{x_0} * \gamma * \bar{e}_{x_0}] = [\gamma]$ ↖ constant path

if σ is a 2-simplex then let y_0, y_1, y_2 be vertices



$$\begin{aligned} \psi(\partial\sigma) &= \psi(\sigma^{(0)} - \sigma^{(1)} + \sigma^{(2)}) = \psi(\sigma^{(2)}) \psi(\sigma^{(0)}) \psi(\sigma^{(1)})^{-1} \\ &= [\gamma_{y_0} * \sigma^{(2)} * \bar{\gamma}_{y_1} * \gamma_{y_1} * \sigma^{(0)} * \bar{\gamma}_{y_2} * \gamma_{y_2} * \bar{\sigma}^{(1)} * \bar{\gamma}_{y_0}] \\ &= [\gamma_{y_0} * \sigma^{(2)} * \sigma^{(0)} * \bar{\sigma}^{(1)} * \gamma_{y_0}] = [e_{x_0}] \end{aligned}$$

↑ since loop bounds disk σ !

So $\text{im } \partial_2 \subset \ker \psi \therefore \psi$ induces a map

$$\psi_* : H_1(X) \rightarrow (\pi_1(X, x_0))^{ab}$$

from above we clearly have $\psi_* \circ \phi_* = \text{id}$

now if $[c] \in H_1(X)$ with $c = \sum_i n_i \sigma_i$

$$\phi_* \circ \psi_*([c]) = \phi_* \left(\left[\sum_i n_i (\gamma_{\sigma_i(0)} * \sigma_i * \bar{\gamma}_{\sigma_i(1)}) \right] \right)$$

↑ concatenate n_i copies

$$= \sum n_i [\gamma_{\sigma_i(0)} * \sigma_i * \bar{\gamma}_{\sigma_i(1)}]$$

↑ concatenate

$$= \sum n_i ([\gamma_{\sigma_i(0)}] + [\sigma_i] - [\gamma_{\sigma_i(1)}])$$

$$= \sum n_i [\sigma_i] = [\sum n_i \sigma_i] = [c]$$

↑ since $\partial c = 0$ for each σ_i , \exists a j st. $\sigma_i(1) = \sigma_j(0)$

$$\text{so } \phi_* \circ \psi_* = \text{id} \quad \square$$

Remark: For any n can similarly define a map

$$\phi_n : \pi_n(X, x_0) \rightarrow H_n(X)$$

and can show for the first k for which $H_k(X) \neq 0$

ϕ_k is an isomorphism if $k > 1$

B Intro to homological algebra and maps on homology

a sequence of abelian groups C_* and maps

$$\partial_n : C_n \rightarrow C_{n-1}$$

↑ denotes general index

is called a chain complex if $\partial_{n-1} \circ \partial_n = 0$ for all n

the homology of the complex is

$$H_n(C_*, \partial) = \ker \partial_n / \text{im } \partial_{n+1}$$

Homological algebra is the study of general chain complexes
 when a definition or theorem is purely about homological algebra
 I'll denote it by (HA) such results are true for any chain complex
 in particular, they will apply to the singular chain groups

defⁿ(HA): given two chain complexes (C_x, ∂) and (C'_x, ∂') a chain map
 is a sequence of homomorphisms $f_n: C_n \rightarrow C'_n$ such that

$$\partial'_n \circ f_n = f_{n-1} \circ \partial_n \quad \dots \rightarrow C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \dots$$

$$\downarrow f_n \quad \partial'_n \quad \downarrow f_{n-1}$$

$$\dots \rightarrow C'_n \xrightarrow{\partial'_n} C'_{n-1} \rightarrow \dots$$

lemma 5 (HA):

A chain map $\{f_n\}: (C_x, \partial) \rightarrow (C'_x, \partial')$ induces homomorphisms
 $(f_n)_* : H_n(C_x, \partial) \rightarrow H_n(C'_x, \partial')$

Proof: note if $[h] \in H_n(C_x, \partial)$, then $\partial_n h = 0$

$$\text{so } \partial'_n(f_n(h)) = f_{n-1}(\partial_n h) = f_{n-1}(0) = 0$$

so $f_n(h)$ gives a class in $H_n(C'_x, \partial')$

$$\text{define } (h_n)_* : H_n(C_x, \partial) \rightarrow H_n(C'_x, \partial')$$

$$[h] \mapsto [f_n(h)]$$

note, $(h_n)_*$ is well-defined since if $[h] = [h']$ then $\exists k$ st.

$$h - h' = \partial_{n+1} k$$

so we have

$$f_n(h) = f_n(h' + \partial_{n+1} k) = f_n(h') + \partial'_{n+1}(f_{n+1} k)$$

$$\text{so } [f_n(h)] = [f_n(h')]$$

$(f_n)_*$ is a homeomorphism since f_n is (exercise) \square

Now for singular homology

if $f: X \rightarrow Y$ is a continuous map, then define

$$f_n: C_n(X) \rightarrow C_n(Y): \sum m_i \sigma_i \mapsto \sum m_i (f \circ \sigma_i)$$

this is clearly a homomorphism and

$$\begin{aligned} f_{n-1}(\partial_n \sigma) &= f_{n-1} \left(\sum_{i=0}^n (-1)^i \sigma^{(i)} \right) = \sum (-1)^i (f \circ \sigma^{(i)}) \\ &= \sum (-1)^i (f \circ \sigma)^{(i)} = \sum (-1)^i f_n(\sigma) \\ &= \partial_n(f_n(\sigma)) \end{aligned}$$

$$\text{so } f_{n-1} \circ \partial_n = \partial_n \circ f_n$$

\therefore by lemma we get

$$f_n: H_n(X) \rightarrow H_n(Y)$$

exercice: 1) $(f \circ g)_n = f_n \circ g_n$

2) $(\text{id}_X)_n = \text{id}_{H_n(X)}$

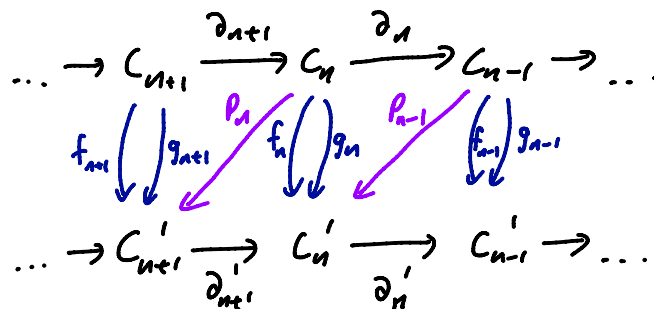
defⁿ (HA):

given chain complexes (C_n, ∂) and (C'_n, ∂')
and chain maps $\{f_n\}$ and $\{g_n\}$ from (C_n, ∂) to (C'_n, ∂')
a chain homotopy between $\{f_n\}$ and $\{g_n\}$ is a sequence of
homeomorphisms

$$P_n: C_n \rightarrow C'_{n+1}$$

such that

$$\partial'_{n+1} P_n + P_{n-1} \circ \partial_n = f_n - g_n$$



lemma 6 (HA):

If there is a chain homotopy between $\{f_n\}$ and $\{g_n\}$

then $(f_n)_* = (g_n)_*: H_n(C_n, \partial) \rightarrow H_n(C'_n, \partial')$

Proof:

$$\text{If } h \in C_n \text{ and } \partial_n h = 0, \text{ then } f_n(h) - g_n(h) = \partial_{n+1}' \circ P_n(h) + P_{n-1}(\partial_n h) \\ = \partial_{n+1}'(P_n(h))$$

$$\text{so } [f_n(h)] = [g_n(h)] \quad \square$$

Th^m 7:

let $f, g: X \rightarrow Y$ be homotopic maps

Then $\{f_n\}$ and $\{g_n\}$ are chain homotopic maps $(C_*(X), \partial)$ to $(C_*(Y), \partial)$

and thus $f_n = g_n: H_n(X) \rightarrow H_n(Y)$

Proof: let $H: X \times [0, 1] \rightarrow Y$ be the homotopy

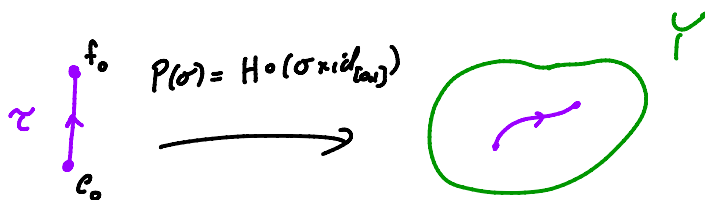
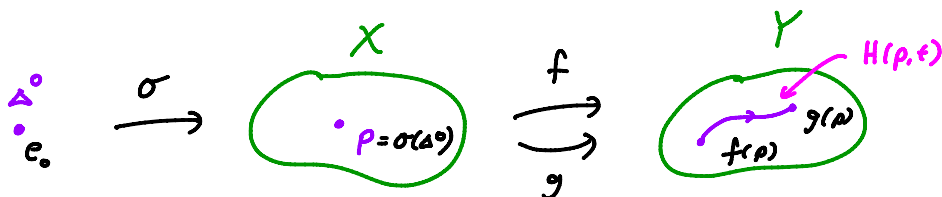
$$\text{so } H(x, 0) = f(x) \text{ and } H(x, 1) = g(x)$$

given a simplex $\sigma: \Delta^n \rightarrow X$

we define $P(\sigma)$ by $H \circ (\sigma \times \text{id}_{[0,1]}) : \Delta^n \times [0, 1] \rightarrow Y$

to make sense of this we need to see $\Delta^n \times [0, 1]$ as a union of $(n+1)$ -simplices

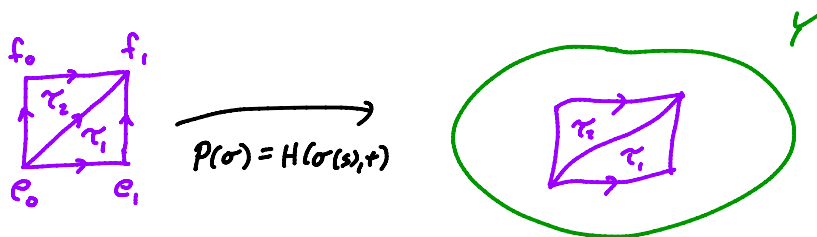
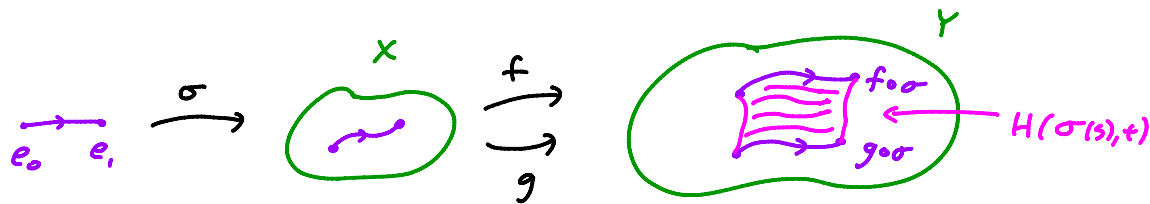
eg $n=0$



$$P(\sigma) = H \circ (\sigma \times \text{id}_{[0,1]}) = H(\sigma(e_0), t)$$

$$\partial P(\sigma) = \partial(H(\sigma(e_0), t)) = H(\sigma(e_0), 1) - H(\sigma(e_0), 0) \\ = g \circ \sigma - f \circ \sigma$$

$n=1$



$$\tau_1 = [e_0, e_1, f_1]$$

$$\tau_0 = [e_0, f_0, f_1]$$

define $P(\sigma) = \overbrace{H_\sigma(\sigma \times id_{[0,1]})}^{H_\sigma} \Big|_{\tau_0} - H_\sigma(\sigma \times id) \Big|_{\tau_1}$

$$\partial(P(\sigma)) = H_\sigma \Big|_{[f_0, f_1]} - \cancel{H_\sigma \Big|_{[e_0, f_1]}} + H_\sigma \Big|_{[e_0, f_0]} - H_\sigma \Big|_{[f_0, f_1]} + \cancel{H_\sigma \Big|_{[e_0, f_1]}} - H_\sigma \Big|_{[e_0, e_1]}$$

$$\downarrow$$

$$= g_0 \sigma + \underbrace{H_\sigma \Big|_{[e_0, f_0]} - H_\sigma \Big|_{[f_0, f_1]}}_{P(\sigma)} - f_0 \sigma$$

$$= g_0 \sigma - f_0 \sigma - P(\sigma)$$

in general let $e_0 \dots e_n$ be the vertices of Δ^n in \mathbb{R}^{n+1}

think of $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$ as $x_{n+2} = 0$

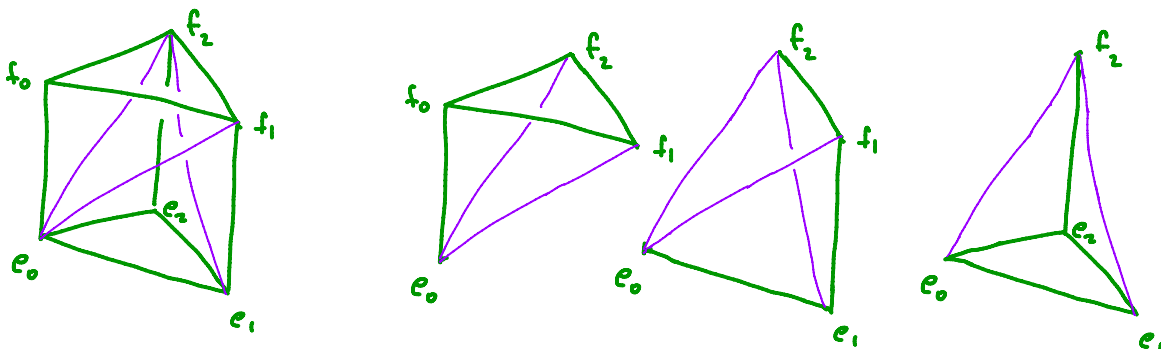
let f_i be the points in \mathbb{R}^{n+2} above e_i with $x_{n+2} \text{ coord} = 1$

so $[e_0 \dots e_n]$ describes $\Delta^n \times \{0\}$ and

$[f_0 \dots f_n]$ " " $\Delta^n \times \{1\}$

and $\Delta^n \times [0, 1]$ is the union of the $(n+1)$ -simplices

$$[e_0, \dots, e_i, f_1, \dots, f_n]$$



now define

$$P: C_n(X) \rightarrow C_{n+1}(Y) \text{ by}$$

$$P(\sigma) = \sum_{i=0}^n (-1)^i H_0(\sigma \times id_{[0,1]}) \Big|_{[e_0 \dots e_i f_i \dots f_n]}$$

$$\begin{aligned} \partial P(\sigma) &= \sum_{j \neq i} (-1)^i (-1)^j H_0(\sigma \times id_{[0,1]}) \Big|_{[e_0 \dots \hat{e}_j \dots e_i f_i \dots f_n]} \\ &\quad + \sum_{i \neq j} (-1)^i (-1)^{j+1} H_0(\sigma \times id_{[0,1]}) \Big|_{[e_0 \dots e_i f_i \dots \hat{f}_j \dots f_n]} \end{aligned}$$

the $i=j$ terms cancel except for

$$[\hat{e}_0 f_0 \dots f_n] = [f_0 \dots f_n] \text{ term which is } f_0 \sigma$$

and

$$[e_0 \dots e_n \hat{f}_n] = [e_0 \dots e_n] \text{ term which is } -f_0 \sigma$$

exercise: the $i \neq j$ terms give $-P(\partial \sigma)$



Cor 8:

If $f: X \rightarrow Y$ is a homotopy equivalence then $f_n: H_n(X) \rightarrow H_n(Y)$ is an isomorphism for all n .

Remark: If X is a contractible space, then

$$H_n(X) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & n \neq 0 \end{cases}$$

C. Relative Homology and Excision

let A be a subspace of X

$$\text{so } C_n(A) \subset C_n(X)$$

note $C_n(A)$ is in the kernel of

$$C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\text{quotient}} C_n(X)/C_n(A)$$

so ∂_n induces a map

$$\partial_n: C_n(X)/C_n(A) \rightarrow C_{n-1}(X)/C_{n-1}(A)$$

define $C_n(X, A) = C_n(X) / C_n(A)$ and

$\partial_n : C_n(X, A) \rightarrow C_{n-1}(X, A)$ as above

clearly $\partial_{n-1} \circ \partial_n = 0$

the relative (singular) homology of (X, A) is

$$H_n(X, A) = \ker \partial_n / \text{im } \partial_{n+1}$$

note: 1) an element in $H_n(X, A)$ is represented by a relative cycle

$$\text{i.e. } \alpha \in C_n(X) \text{ s.t. } \partial_n \alpha \in C_{n-1}(A)$$

2) a relative cycle α is trivial in $H_n(X, A)$ if it is a relative boundary

$$\text{i.e. } \exists \beta \in C_{n+1}(X) \text{ and } \gamma \in C_n(A) \text{ s.t.}$$

$$\alpha = \partial\beta + \gamma$$

Recall a sequence of homomorphisms

$$A \xrightarrow{\phi} B \xrightarrow{\psi} C$$

is called exact at B if $\text{im } \phi = \ker \psi$

a longer sequence is exact if it is exact at every group

lemma 9 (HA):

If (A_n, ∂_A) , (B_n, ∂_B) , and (C_n, ∂_C) are 3 chain complexes and $\{\phi_n\}: (A_n, \partial_A) \rightarrow (B_n, \partial_B)$ and $\{\psi_n\}: (B_n, \partial_B) \rightarrow (C_n, \partial_C)$ are chain maps such that

$$0 \rightarrow A_n \xrightarrow{\phi_n} B_n \xrightarrow{\psi_n} C_n \rightarrow 0$$

is exact $\forall n$

then there is a long exact sequence

$$\dots \rightarrow H_n(A_n, \partial_A) \xrightarrow{(\phi_n)_*} H_n(B_n, \partial_B) \xrightarrow{(\psi_n)_*} H_n(C_n, \partial_C) \xrightarrow{\partial_n} H_{n-1}(A_n, \partial_A) \rightarrow \dots$$

Some times this is written as an exact triangle

$$\begin{array}{ccc} H_n(A, \partial_A) & \xrightarrow{\phi_*} & H_n(B, \partial_B) \\ & \searrow \partial_* & \swarrow \psi_* \\ & H_{n-1}(C, \partial_C) & \end{array}$$

ϕ_*, ψ_* preserve degree
 ∂_* reduces degree by 1

before proving this we not an important consequence

Th^m 10:

let A be a subspace of X

1) let $i: A \rightarrow X$ be the inclusion map and $q_n: C_n(X) \rightarrow C_n(X, A)$ the quotient map

Then

$$0 \rightarrow C_n(A) \xrightarrow{i_n} C_n(X) \xrightarrow{q_n} C_n(X, A) \rightarrow 0$$

is exact

2) So \exists a long exact sequence

$$\dots \rightarrow H_n(A) \xrightarrow{i_n} H_n(X) \xrightarrow{q_n} H_n(X, A) \xrightarrow{\partial_n} H_{n-1}(A) \rightarrow \dots$$

called long exact sequence of a pair.

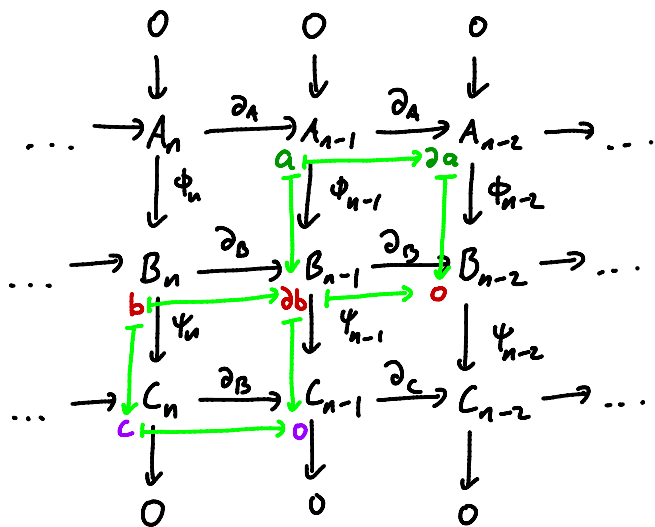
Proof: 2) follows from 1) and lemma 9

- 1) clearly $i_n: C_n(A) \rightarrow C_n(X)$ is injective
- and $q_n: C_n(X) \rightarrow C_n(X, A)$ is surjective
- and $\ker q_n = \text{im } i_n$ \square

exercise: verify that ∂_n in the exact sequence is the map $h \in H_n(X, A)$, choose $\alpha \in C_n(X)$ s.t. $\partial_n \alpha \in C_{n-1}(A)$ and $\alpha \in h$

then $\partial_n h = [\partial_n \alpha] \in H_{n-1}(A)$

Proof of lemma 9: consider



given $c \in C_n$ with $\partial_c c = 0$

$\exists b \in B_n$ s.t. $\psi_n(b) = c$

note: $\psi_{n-1}(\partial_B b) = \partial_C(\psi_n(b)) = \partial_C(c) = 0$

so $\partial_B b \in \ker \psi_{n-1} = \text{im } \phi_{n-1}$

and $\exists a \in A_{n-1}$ s.t. $\phi_{n-1}(a) = \partial_B b$

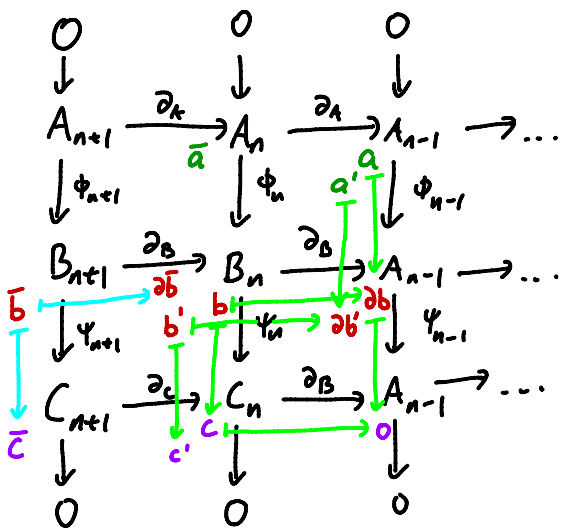
note: $\phi_{n-2}(\partial_A a) = \partial_B(\phi_{n-1}(a)) = \partial_B^2 b = 0$

since ϕ_{n-2} injective $\partial_A a = 0$

define: $\partial: H_n(C, \partial) \rightarrow H_{n-1}(A, \partial_A)$

$[c] \mapsto [a]$ where a is constructed above

Claim: $\partial[c]$ is well-defined



note: only 2 choices: ① $c \in [b]$ and ② b s.t. $\Psi_n(b) = c$

now given $c, c' \in [c]$

$$\exists \bar{c} \in C_{n+1} \text{ s.t. } \partial_C \bar{c} = c - c'$$

choose any $b, b' \in B_n$ and $\bar{b} \in B_{n+1}$ s.t.

$$\Psi_n(b) = c, \Psi_n(b') = c', \text{ and } \Psi_{n+1}(\bar{b}) = \bar{c}$$

$$\text{so } \Psi_n(\partial_B \bar{b} - b + b') = \partial_C \Psi_{n+1}(\bar{b}) - c + c' = 0$$

and hence $\exists \bar{a} \in A_n$ s.t. $\phi_n(\bar{a}) = \partial_B \bar{b} - b + b'$

as above $\exists! a, a' \in A_{n-1}$ s.t. $\phi_{n-1}(a) = \partial_B b$ and $\phi_{n-1}(a') = \partial_B b'$

$$\begin{aligned} \text{now } \phi_{n-1}(\partial_A \bar{a} + a - a') &= \partial_B \phi_n(\bar{a}) + \partial_B b - \partial_B b' \\ &= \partial_B (\partial_B \bar{b} - b + b') + \partial_B b - \partial_B b' = 0 \end{aligned}$$

$$\phi_{n-1} \text{ injective} \Rightarrow a' = a + \partial_A \bar{a}$$

and hence $[a] = [a']$ i.e. $\partial[c]$ well-defined

exercise: 1) show ∂ a homomorphism

$$2) \text{im}(\phi_n)_* = \ker(\Psi_n)_*$$

$$3) \text{im}(\Psi_n)_* = \ker \partial_n$$

$$4) \text{im} \partial_n = \ker(\phi_{n-1})_*$$



example: $H_n(X, x_0) = \tilde{H}_n(X)$

reduced homology

$$\text{indeed } H_n(x_0) \rightarrow H_n(X) \rightarrow H_n(X, x_0) \rightarrow H_{n-1}(x_0)$$

for $n \geq 2$



$$\text{so } H_n(X) = H_n(X, x_0)$$

and $H_1(x_0) \rightarrow H_1(X) \rightarrow H_1(X, x_0) \rightarrow H_0(x_0) \rightarrow H_0(X) \rightarrow H_0(x, x_0) \rightarrow 0$

$$\begin{array}{ccccccc} \parallel & & & & & & \\ 0 & \rightarrow & H_1(X) & \rightarrow & H_1(X, x_0) & \rightarrow & \mathbb{Z} \rightarrow \bigoplus_k \mathbb{Z} \rightarrow H_0(X, x_0) \rightarrow 0 \end{array}$$

path components
isomorphism onto \mathbb{Z} given by path component of X containing x_0

$\therefore 0$ map

so $H_1(X, x_0) = H_1(X)$ and

$$H_0(X, x_0) = \bigoplus_{k-1} \mathbb{Z}$$

exercises:

1) If $f: (X, A) \rightarrow (Y, B)$ is continuous then

$$f_*: H_n(X, A) \rightarrow H_n(Y, B)$$

2) if $f, g: (X, A) \rightarrow (Y, B)$ are homotopic through maps taking A to B

then $f_* = g_*$

3) $A \subset B \subset X$ then you get a long exact sequence

$$\dots \rightarrow H_n(B, A) \rightarrow H_n(X, A) \rightarrow H_n(X, B) \rightarrow H_{n-1}(B, A) \rightarrow \dots$$

Th^m 11 (Excision):

let $Z \subset A$ be subspaces of X

assume $\bar{Z} \subset \text{int } A$

Then the inclusion map

$$\gamma: (X-Z, A-Z) \rightarrow (X, A)$$

induces an isomorphism on homology

$$\gamma_*: H_n(X-Z, A-Z) \rightarrow H_n(X, A)$$

We give the proof later

a pair $A \subset X$ is called good if A is non-empty, closed, and has a neighborhood U in X s.t. A is a deformation retract of U

(i.e. $\exists H: U \times [0, 1] \rightarrow U$ s.t. $H(x, 0) = x, H(x, 1) \in A \quad \forall x \in U$, and

$$H(x, t) = x \quad \forall x \in A.)$$

examples: if A is a submanifold of a manifold X , then (X, A) a good pair
if X is built from A by attaching cells, then (X, A) is a good pair

Th¹²:

If (X, A) is a good pair, then the quotient map

$$q: (X, A) \rightarrow (X/A, A/A)$$

induces an isomorphism

$$q_*: H_n(X, A) \rightarrow H_n(X/A, A/A) \cong H_n(X/A, pt) \cong \tilde{H}_n(X/A)$$

Proof: let U be a neighborhood of A that def retracts to A

we have

$$\begin{array}{ccccc} H_n(X, A) & \rightarrow & H_n(X, U) & \xleftarrow{\cong \text{ by excision (Th } \# 11)} & H_n(X-A, U-A) \\ \downarrow q_* & \circ & \downarrow q_* & \circ & \downarrow q_* \\ H_n(X/A, A/A) & \rightarrow & H_n(X/A, U/A) & \xleftarrow{\cong} & H_n(X/A - A/A, U/A - A/A) \end{array}$$

note: $q: (X-A, U-A) \rightarrow (X/A - A/A, U/A - A/A)$ a homeomorphism! (the "identity map")

so right most q_* an isomorphism

so middle q_* an isomorphism by excision

now we have $(A, A) \xrightarrow{i} (U, A) \xrightarrow{h_1} (A, A)$ where $h_1(x) = H(x, 1)$ and

and $h_1 \circ i = id_{(A, A)}$

$i \circ h_1 \simeq id_{(U, A)}$ by h_t

H is def. retraction U to A

so $z_*: H_n(A, A) \rightarrow H_n(U, A)$ an isomorphism

thus the long exact sequence of $A \subset U \subset X$ gives

$$\begin{array}{ccccccc} H_n(U, A) & \rightarrow & H_n(X, A) & \rightarrow & H_n(X, U) & \rightarrow & H_{n-1}(U, A) \\ \parallel & & & & & & \parallel \\ H_n(A, A) & & & & & & H_{n-1}(A, A) \\ \parallel & & & & & & \parallel \\ 0 & & & & & & 0 \end{array}$$

and $H_n(X, A) \rightarrow H_n(X, U)$ an isomorphism.

we also know U/A deformation retracts to A/A (H induces def retract)
so the same argument shows

$$H_n(X/A, A/A) \rightarrow H_n(X/A, U/A) \text{ is an isomorphism}$$

\therefore left most q_* above is an isomorphism as claimed \square

Prop 13:

$$H_k(S^n) = \begin{cases} \mathbb{Z} & k=0, n \\ 0 & k \neq 0, n \end{cases}$$

and

$$H_k(D^n, \partial D^n) = \begin{cases} \mathbb{Z} & k=n \\ 0 & k \neq n \end{cases}$$

Proof: note

$$H_k(S^0) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & k=0 \\ 0 & k \neq 0 \end{cases}$$

and

$$H_k(D^n) = \begin{cases} \mathbb{Z} & k=0 \\ 0 & k \neq 0 \end{cases}$$

n=1 case: Long exact sequence of (D^1, S^0) (note $D^1/S^0 \cong S^1$)

$$H_k(S^0) \rightarrow H_k(D^1) \rightarrow H_k(D^1, S^0) \rightarrow H_{k-1}(S^0)$$

$\begin{matrix} \text{"} & & \text{"} & & \text{"} \\ 0 & & 0 & & 0 \end{matrix}$

so $H_k(S^1) \cong H_k(D^1/S^0) \cong H_k(D^1, S^0) = 0$ for $k \geq 2$

$$H_1(S^0) \rightarrow H_1(D^1) \rightarrow H_1(D^1, S^0) \rightarrow H_0(S^0) \rightarrow H_0(D^1) \rightarrow H_0(D^1, S^0)$$

$\begin{matrix} \text{"} & \text{"} & & \text{"} & \text{"} & \text{"} \\ 0 & 0 & & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z} \\ & & & \text{"} & \text{"} & \text{"} \\ & & & & & \mathbb{Z} \\ & & & & & \text{"} \\ & & & & & 0 \end{matrix}$

$$\Rightarrow H_1(S^1) \cong H_1(D^1, S^0) \cong \mathbb{Z}$$

induction: assume Prop is true for S^{n-1} ($n \geq 1$)

Consider (D^n, S^{n-1}) :

$$H_n(D^n) \rightarrow H_n(D^n, S^{n-1}) \rightarrow H_{n-1}(S^{n-1}) \rightarrow H_{n-1}(D^n)$$

$\begin{matrix} \text{"} & & \text{"} & & \text{"} \\ 0 & & \mathbb{Z} & & 0 \end{matrix}$

so $H_n(S^n) \cong H_n(D^n, S^{n-1}) \cong \mathbb{Z}$

for $k \neq n, 0, 1$

$$H_k(D^n) \rightarrow H_k(D^n, S^{n-1}) \rightarrow H_{k-1}(S^{n-1}) \rightarrow H_{k-1}(D^n)$$

$\begin{matrix} \text{"} & & \text{"} & & \text{"} \\ 0 & & 0 & & 0 \end{matrix}$

so $H_k(S^n) \cong H_k(D^n, S^{n-1}) = 0$ $k \neq n, 1$.

now

$$H_1(D^n) \rightarrow H_1(D^n, S^{n-1}) \rightarrow H_0(S^{n-1}) \rightarrow H_0(D^n) \rightarrow H_0(D^n, S^{n-1})$$

$\begin{matrix} \text{"} & & \text{"} & & \text{"} & & \text{"} \\ 0 & & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{matrix}$

so $H_1(S^n) = H_1(D^n, S^{n-1}) = 0$



Cor 14:

∂D^n is not a retract of D^n and
any map $f: D^n \rightarrow D^n$ has a fixed point.

Proof:

If $r: D^n \rightarrow \partial D^n$ is a retraction then $r \circ i = id_{\partial D^n}$ where $i: \partial D^n \rightarrow D^n$ is inclusion

thus $r_* \circ i_*: H_{n-1}(\partial D^n) \rightarrow H_{n-1}(\partial D^n)$ is an isomorphism
 $\cong \cong$

but $r_*: H_{n-1}(\partial D^n) \rightarrow H_{n-1}(D^n) = 0$ the trivial map! \otimes

$\therefore r$ does not exist!

the second statement follows from the first as in proof of Cor I. 14 \square

Cor 15:

If $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open sets that are homeomorphic
then $n = m$

Remark: this is called "invariance of domain" and implies that any
 n manifold is not homeomorphic to an m manifold for $n \neq m$

Proof: for any $x \in U$, we have $\mathbb{R}^n - U \subset \mathbb{R}^n - \{x\} \subset \mathbb{R}^n$, so excision says

$$H_k(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong H_k(\mathbb{R}^n - (\mathbb{R}^n - U), (\mathbb{R}^n - \{x\}) - (\mathbb{R}^n - U)) = H_k(U, U - \{x\})$$

the exact sequence for $(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ says

$$H_k(\mathbb{R}^n) \rightarrow H_k(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \rightarrow H_{k-1}(\mathbb{R}^n - \{x\}) \rightarrow H_{k-1}(\mathbb{R}^n)$$

$\cong \cong$ \cong \cong

$$\text{so } H_k(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong H_{k-1}(\mathbb{R}^n - \{x\}) \cong H_{k-1}(S^{n-1})$$

$$\text{and } H_k(U, U - \{x\}) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n \end{cases} \quad \mathbb{R}^n - \{x\} \cong S^{n-1}$$

$$\text{similarly } H_k(V, V - \{y\}) \cong \begin{cases} \mathbb{Z} & k = m \\ 0 & k \neq m \end{cases} \quad \forall y \in V$$

if $h: U \rightarrow V$ a homeomorphism then $H_k(U, U - \{x\}) \cong H_k(V, V - \{h(x)\}) \quad \forall k$

$\therefore m = n$ \square

later it will be useful to know the generators of $H_n(S^n)$ and $H_n(D^n, \partial D^n)$

Prop 16:

- 1) We can identify $(D^n, \partial D^n)$ with $(\Delta^n, \partial \Delta^n)$
under this identification

$$H_n(D^n, \partial D^n) \cong \mathbb{Z}$$

is generated by the identity map $\Delta^n \rightarrow \Delta^n$

- 2) We can identify S^n with $\Delta_1^n \cup \Delta_2^n$ with $\partial \Delta_1^n$ glued to $\partial \Delta_2^n$
by a homeomorphism that preserves
the order of the vertices and is
the identity on faces



if $f_1: \Delta_1^n \rightarrow S^n$ is the inclusion map

then $f_1 - f_2$ is the generator of $H^n(S^n) \cong \mathbb{Z}$

Proof: 1) we induct on n

$n=0$: $H_0(D^0, \partial D^0) = H_0(D^0, \emptyset) = \mathbb{Z}$

\uparrow pt

$H_0(D^0)$ is the free abelian group gen. by path components (by Th 2.3)

so generator is $D^0 = \Delta^0$

induction: assume we know $H_{n-1}(D^{n-1}, \partial D^{n-1})$ generated by the inclusion

$$f: \Delta^{n-1} \rightarrow \Delta^{n-1} \cong D^{n-1}$$

clearly the inclusion $g: \Delta^n \rightarrow \Delta^n$ is a relative cycle ($\partial_n g \in C_{n-1}(\partial \Delta^n)$)

so $\partial g = 0$ in $C_{n-1}(D^n, \partial D^n)$

let Λ be the union of all but one of the $(n-1)$ -dim'l faces of Δ^n



consider the long exact sequence for $(\Delta^n, \partial \Delta^n, \Lambda)$

$$\begin{array}{ccccccc}
 H_n(\Delta^n, \Lambda) & \rightarrow & H_n(\Delta^n, \partial\Delta^n) & \xrightarrow{\partial} & H_{n-1}(\partial\Delta^n, \Lambda) & \rightarrow & H_{n-1}(\Delta^n, \partial\Delta^n) \\
 \cong & & & & & & \cong \text{ same} \\
 \tilde{H}_n(\Delta^n/\Lambda) & & & & & & \\
 \cong & \xrightarrow{\cong \Delta^n \cong pt} & & & & & \\
 0 & & & & & &
 \end{array}$$

so ∂ is an isomorphism

note $\partial\Delta^n = \Lambda \cup \Delta^{n-1}$,

$(\partial\Delta^n, \Lambda)$ a good pair, and

$$\partial\Delta^n/\Lambda \cong \Delta^{n-1}/\partial\Delta^{n-1} \quad \text{induced by inclusion } i: \Delta^{n-1} \rightarrow \partial\Delta^n$$

$$\text{so } H_{n-1}(\partial\Delta^n, \Lambda) \xrightarrow[\cong]{\Psi} H_{n-1}(\Delta^{n-1}/\partial\Delta^{n-1}) \cong H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$$

by induction the inclusion of Δ^{n-1} into Δ^{n-1} generates $H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$

\therefore inclusion Δ^{n-1} into $\partial\Delta^n$ generates $H_{n-1}(\partial\Delta^n, \Lambda)$

now $\partial[g] = [\partial g] = [\partial\Delta^n]$ in $H_{n-1}(\partial\Delta^n, \Lambda)$

but $[\partial\Delta^n] = [\Delta^{n-1}]$ in $H_{n-1}(\partial\Delta^n, \Lambda)$

$\therefore \partial[g]$ is a generator of $H_{n-1}(\partial\Delta^n, \Lambda)$

and $[g]$ a generator of $H_n(\Delta^n, \partial\Delta^n)$ since ∂ an isomorphism

now for S^n : note $S^n/\Delta_2^n \cong \Delta_1^n/\partial\Delta_1^n$

this homeomorphism comes from

$$\begin{array}{ccccc}
 \Delta_1^n & \xrightarrow{\text{inc.}} & S^n & \xrightarrow{\text{quot.}} & S^n/\Delta_2^n \\
 & & & & \uparrow \\
 & & & & f
 \end{array}$$

this descends to $h: \Delta_1^n/\partial\Delta_1^n \rightarrow S^n/\Delta_2^n$ a homeomorphism

\therefore we get an isomorphism $h_*: H_n(\Delta_1^n, \partial\Delta_1^n) \rightarrow H_n(S^n, \Delta_2^n)$

so Δ_1^n maps to a generator $[\Delta_1^n]$ of $H_n(S^n, \Delta_2^n)$

$$\text{note: } H_n(\Delta_2^n) \rightarrow H_n(S^n) \xrightarrow{\cong} H_n(S^n, \Delta_2^n) \rightarrow H_{n-1}(\Delta_2^n)$$

induced by inclusion $i: (S^n, \emptyset) \rightarrow (S^n, \Delta_2^n)$

so $H_n(S^n) \cong \mathbb{Z}$ and it is generated by a cycle in $C_n(S^n)$ that

maps by i to Δ_1^n

now $\sigma = \Delta_1^1 - \Delta_2^1$ is a cycle (exercise if not obvious)

and $\sigma = \Delta_1^1$ in $C_n(S^n) / C_n(\Delta_2^1)$

$\therefore \tau_*([\sigma]) = [\Delta_1^1] \in H_n(S^1, \Delta_2^1)$

and hence $[\sigma]$ generates $H_n(S^n)$ \square

When making computations it is useful to know the long exact sequences "respect" maps between spaces. This is called naturality.

Th^m 17:

If $f: (X, A) \rightarrow (Y, B)$ is a map of pairs, then the following diagram is commutative

$$\begin{array}{ccccccc} \dots & \rightarrow & H_n(A) & \rightarrow & H_n(X) & \rightarrow & H_n(X, A) & \rightarrow & H_{n-1}(A) & \rightarrow & \dots \\ & & \downarrow f_* & \circ & \downarrow f_* & \circ & \downarrow f_* & \circ & \downarrow f_* & & \\ \dots & \rightarrow & H_n(B) & \rightarrow & H_n(Y) & \rightarrow & H_n(Y, B) & \rightarrow & H_{n-1}(B) & \rightarrow & \dots \end{array}$$

similarly for the long exact sequence of a triple.

Proof: note

$$\begin{array}{ccccccc} 0 & \rightarrow & C_n(A) & \rightarrow & C_n(X) & \rightarrow & C_n(X, A) & \rightarrow & 0 \\ & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \\ 0 & \rightarrow & C_n(B) & \rightarrow & C_n(Y) & \rightarrow & C_n(Y, B) & \rightarrow & 0 \end{array}$$

is clearly commutative so result follows from homological algebra lemma below \square

lemma 18 (HA):

If we have chain complexes and chain maps s.t.

$$\begin{array}{ccccccc} 0 & \rightarrow & A_n & \xrightarrow{i} & B_n & \xrightarrow{j} & C_n & \rightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \rightarrow & A'_n & \xrightarrow{i'} & B'_n & \xrightarrow{j'} & C'_n & \rightarrow & 0 \end{array}$$

is commutative and rows are exact

then

$$\begin{array}{ccccccc} \dots & \rightarrow & H_n(A_n) & \xrightarrow{i_n} & H_n(B_n) & \xrightarrow{j_n} & H_n(C_n) & \xrightarrow{\partial} & H_{n-1}(A_n) & \rightarrow & \dots \\ & & \downarrow \alpha_n & & \downarrow \beta_n & & \downarrow \gamma_n & & & & \\ \dots & \rightarrow & H_n(A'_n) & \xrightarrow{i'_n} & H_n(B'_n) & \xrightarrow{j'_n} & H_n(C'_n) & \xrightarrow{\partial} & H_{n-1}(A'_n) & \rightarrow & \dots \end{array}$$

is commutative

Proof: since $\beta \circ i = i' \circ \alpha \Rightarrow \beta_* \circ i_* = i'_* \circ \alpha_*$

similarly for $\gamma_* \circ j_* = j'_* \circ \beta_*$

now recall $\partial [c] = [a]$ where $a \in A_{n-1}$ s.t.

$$i(a) = \partial b \text{ for some } b \in B_n \text{ s.t. } j(b) = c$$

$$\text{so } \partial [\gamma(c)] = [\alpha(a)] \text{ since } \gamma(c) = \gamma(j(b)) = j'(b)$$

$$\text{and } i'(\alpha(a)) = \beta(i(a)) = \beta(\partial b) = \partial \beta(b)$$

β chain map

$$\therefore \partial \circ \gamma_* [c] = \alpha_* \circ \partial [c] \quad \forall [c]$$

Here is a long exact sequence that generalizes Van Kampen to homology theory

Thm 19 (Mayer-Vietoris):

let $A, B \subset X$ be subspaces s.t. $X = (\text{int } A) \cup (\text{int } B)$

let

$$\begin{array}{ccccc} A \cap B & \xrightarrow{i_A} & A & \xrightarrow{j_A} & X \\ & \searrow i_B & & \nearrow j_B & \\ & & B & & \end{array} \quad \text{be inclusion maps}$$

$$\text{then } \dots \rightarrow H_n(A \cap B) \xrightarrow{\phi} H_n(A) \oplus H_n(B) \xrightarrow{\psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \rightarrow \dots$$

is exact where $\phi = (i_A)_* \oplus (i_B)_*$

$$\psi([a], [b]) = (j_A)_*[a] - (j_B)_*[b] \text{ and}$$

$$\partial [z] = [\partial a] \text{ where } z = a + b \text{ for } a \in C_n(A) \text{ and } b \in C_n(B)$$

example: $T^2 = S^1 \times S^1$

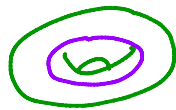


$$A = \text{inner circle} = (0, \frac{\pi}{2}) \times S^1 \cong S^1$$

$$H_n(A) = H_n(B) \cong \begin{cases} \mathbb{Z} & n=0,1 \\ 0 & n \neq 0,1 \end{cases}$$

$$B = T^2 - C \cong S^1$$

where



$$C = \{\frac{\pi}{2}\} \times S^1$$

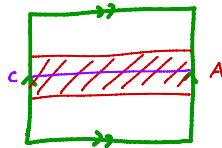
$$H_n(A \cap B) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n=0,1 \\ 0 & n \neq 0,1 \end{cases}$$

$$\begin{array}{cccccccccccc}
 H_2(A) \oplus H_2(B) & \rightarrow & H_2(T^2) & \rightarrow & H_1(A \cap B) & \rightarrow & H_1(A) \oplus H_1(B) & \rightarrow & H_1(T^2) & \rightarrow & H_0(A \cap B) & \rightarrow & H_0(A) \oplus H_0(B) & \rightarrow & H_0(T^2) & \rightarrow & 0 \\
 \parallel & & & & \parallel & & \parallel & & \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\
 0 & \longrightarrow & H_2(T^2) & \xrightarrow{\phi_1} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\psi_1} & H_1(T^2) & \xrightarrow{\partial} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\phi_0} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\psi_0} & \mathbb{Z} & \oplus & \mathbb{Z} & \longrightarrow & 0
 \end{array}$$

since path connected

note $\phi_0(1,0) = (1,1)$
 $\phi_0(0,1) = (1,1)$
 $\phi_1(1,0) = (1,1)$
 $\phi_1(0,1) = (1,1)$

each path component of $A \cap B$
maps into a path component of A by τ_A and B by τ_B



$\Delta_1 - \Delta_2$ generates one factor of $H_1(A \cap B)$
and $H_1(A)$
similarly for other component of $A \cap B$

so $\ker \phi_1 \cong \mathbb{Z}$ generated by $(1,-1)$

$$\therefore H_2(T^2) \cong \mathbb{Z}$$

$\ker \psi_1 = \text{im } \phi_1 \cong \mathbb{Z}$ generated by $(1,1)$

$$\therefore \text{im } \psi_1 = \mathbb{Z} \oplus \mathbb{Z} / \ker \psi_1 \cong \mathbb{Z}$$

$\text{im } \partial \cong \ker \phi_0 \cong \mathbb{Z}$ generated by $(1,-1)$

$$\text{so } \mathbb{Z} \cong \text{im } \partial \cong H_1(T^2) / \ker \partial \cong H_1(T^2) / \text{im } \psi_1 \cong H_1(T^2) / \mathbb{Z}$$

Claim: $H_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$

let $a+K$ generate $\mathbb{Z} \cong H_1(T^2)/K$ where $K \cong \mathbb{Z}$

if $e \in H_1(T^2)$ then $e+K = na+K$ some $n \in \mathbb{Z}$

now $e-na \in K$ so if b generates K , then

$$e-na = mb \text{ some } m \in \mathbb{Z}$$

so each $e \in H_1(T^2)$ is $na+mb$

you can easily argue that $na+mb = n'a+m'b \Leftrightarrow n=n', m=m'$

exercise: $H_n(T^2) = 0 \forall n \geq 3$

$$\text{so } H_n(T^2) \cong \begin{cases} \mathbb{Z} & n=0,2 \\ \mathbb{Z} \oplus \mathbb{Z} & n=1 \\ 0 & \text{otherwise} \end{cases}$$

to establish Mayer-Vietoris we need

lemma 20(HA):

given two long exact sequences and maps between them as below so that the diagram commutes

$$\begin{array}{ccccccccccc} \dots & \rightarrow & C_{n+1} & \rightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \xrightarrow{h_n} & A_{n-1} & \rightarrow & B_{n-1} & \rightarrow & \dots \\ & & \downarrow \delta_{n+1}' & & \downarrow \alpha_n & & \downarrow \beta_n & & \downarrow \gamma_n & & \downarrow \alpha_{n-1}' & & \downarrow \beta_{n-1}' & & \\ \dots & \rightarrow & C_{n+1}' & \rightarrow & A_n' & \xrightarrow{f_n'} & B_n' & \xrightarrow{g_n'} & C_n' & \xrightarrow{h_n'} & A_{n-1}' & \rightarrow & B_{n-1}' & \rightarrow & \dots \end{array}$$

If the γ_n are isomorphisms, then

$$\dots \rightarrow A_n \xrightarrow{\Phi_n} A_n' \oplus B_n' \xrightarrow{\Psi_n} B_n' \xrightarrow{\Gamma_n} A_{n-1} \rightarrow \dots$$

is exact where $\Phi_n(a) = (\alpha_n(a), f_n(a))$,

$\Psi_n(a', b) = \beta_n(b) - f_n'(a)$, and

$\Gamma_n(b') = h_n \circ \gamma_n^{-1} \circ g_n'(b')$

Proof: easy diagram chase

for example lets check exactness at B_n'

$$\begin{aligned} \Gamma_n \circ \Psi_n(a', b) &= h_n \circ \gamma_n^{-1} \circ g_n'(\beta_n(b) - f_n'(a)) \\ &= h_n \circ \gamma_n^{-1}(\gamma_n \circ g_n(b)) = h_n \circ g_n(b) = 0 \end{aligned}$$

(since $g_n' \circ f_n' = 0$)

so $\text{im } \Psi_n \subset \ker \Gamma_n$

now if $b' \in \ker \Gamma_n$ then $\gamma_n^{-1} \circ g_n'(b') \in \ker h_n$

so $\exists b \in B_n$ s.t. $g_n(b) = \gamma_n^{-1} \circ g_n'(b')$

i.e. $g_n'(b') = \gamma_n \circ g_n(b) = g_n' \circ \beta_n(b)$

thus $g_n'(\beta_n(b) - b') = 0$

and so $\exists a' \in A_n'$ s.t. $f_n'(a') = \beta_n(b) - b'$

$\therefore b' = -f_n'(a') + \beta_n(b) = \Psi_n(a', b)$

thus $\text{im } \Psi_n = \ker \Gamma_n$

exercise: check other cases \square

Proof of Th^m 19 (Mayer-Vietoris):

Consider the long exact sequences of pairs $(A, A \cap B)$ and (X, B)

$$\begin{array}{ccccccc} H_n(A \cap B) & \xrightarrow{(i_A)_*} & H_n(A) & \xrightarrow{(k_A)_*} & H_n(A, A \cap B) & \rightarrow & H_{n-1}(A \cap B) \\ & \downarrow (i_B)_* & \downarrow (i_A)_* & & \downarrow I_* & & \downarrow \\ H_n(B) & \xrightarrow{(j_B)_*} & H_n(X) & \xrightarrow{(k_B)_*} & H_n(X, B) & \rightarrow & H_{n-1}(B) \end{array}$$

where k_A, k_B and I are obvious inclusions

note: $A/A \cap B \cong X/B$ (exercise prove this. almost obvious)

so if $(A, A \cap B)$ and (X, B) are good pairs then I_* is

$$\begin{array}{ccc} \text{an isomorphism since } H_n(A, A \cap B) \cong \tilde{H}_n(A/A \cap B) & & \\ \downarrow \Phi_* \circ & \downarrow \Phi_* \cong & \\ H_n(X, B) \cong H_n(X/B) & & \end{array}$$

If pairs not good still expect Φ is

an isomorphism, and it is (see book, or try to prove after we prove excision)

the result now follows from lemma (check if not obvious) 

Finally back to proof of excision

Proof Th^m 11 (Excision):

recall the set up: let $Z \subset A$ be subspaces of X with $\bar{Z} \subset \text{int } A$
we need to show the inclusion map

$$\gamma: (X-Z, A-Z) \rightarrow (X, A)$$

induces an isomorphism on homology

$$\gamma_*: H_n(X-Z, A-Z) \rightarrow H_n(X, A)$$

Main idea: for any n and any $\alpha = \sum_{i=1}^k m_i \sigma_i \in C_n(X, A) = C_n(X)/C_n(A)$

we can find $\beta \in C_{n+1}(X, A)$ such that

$$(*) \quad \alpha + \partial\beta = \sum_{i=1}^l m_i \tau_i$$

where $\text{im } \tau_i \subset X-Z$ or $\text{im } \tau_i \subset A \quad \forall i$

we prove \otimes later but see how it implies theorem.

take $[\alpha] \in H_n(X, A)$

$\otimes \Rightarrow \exists \alpha' \in [\alpha]$ s.t. $\alpha' = \sum_i m_i \tau_i$ as in \otimes

$$\text{let } \alpha'' = \sum_{\substack{i \text{ s.t.} \\ \text{im } \tau_i \subset A}} m_i \tau_i$$

note: 1) $\alpha'' = \alpha'$ in $C_n(X)/C_n(A) = C_n(X, A)$

2) $\partial \alpha'' \in C_n(A)$ in fact in $C_n(A-Z)$

(since $-\partial \alpha'' = \partial(\alpha' - \alpha'') - \partial \alpha'$ so $\partial \alpha'' \subset A \cap (X-Z)$)

3) $\alpha'' \in C_n(X-Z)/C_n(A-Z)$

$\therefore \alpha''$ defines an element $[\alpha''] \in H_n(X-Z, A-Z)$

clearly $\iota_*([\alpha'']) = [\alpha''] = [\alpha'] = [\alpha]$

so ι_* is onto

now suppose $[\alpha] \in H_n(X-Z, A-Z)$ and $\iota_*([\alpha]) = 0$

so $\iota_0 \alpha = \alpha = \partial \beta$ some $\beta \in C_{n+1}(X, A)$

$\otimes \Rightarrow \exists \gamma \in C_{n+2}(X, A)$ s.t. $\beta + \partial \gamma = \sum m_i \tau_i$ with τ_i as in \otimes

clearly $\alpha = \partial(\beta + \partial \gamma)$

$$\text{let } \beta' = \sum_{\substack{i \text{ s.t.} \\ \text{im } \tau_i \subset A}} m_i \tau_i$$

note: 1) $\beta' \in C_n(X-Z)$ and

2) $\partial \beta' = \partial \beta$ - terms with image in A

so $\partial \beta' = \alpha$ in $C_n(X)/C_n(A)$

but $\beta' \in C_n(X-Z)$

$\therefore [\alpha] = 0$ in $H_n(X-Z, A-Z)$

and ι_* injective ✓

now to prove \otimes :

need barycentric subdivision of a simplex

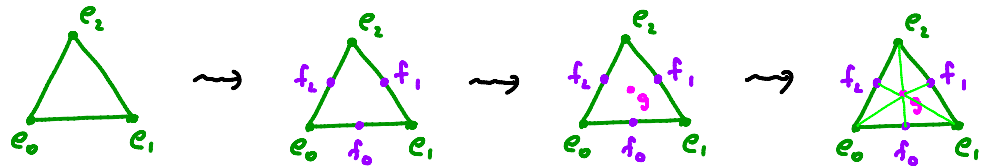
subdivision of 0-simplex: do nothing

" " 1-simplex: break simplex into 2 equal pieces



$$[e_0, e_1] = [e_0, f] \cup [f, e_1]$$

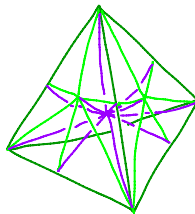
subdivision of 2-simplex: subdivide faces
add center point
add edges to all vertices



$$[e_0, e_1, e_2] = [g, e_0, f_0] \cup [g, f_0, e_1] \cup [g, e_1, f_1] \cup \dots$$

inductively subdivision of n-simplex: subdivide faces
add center point
add edges to all vertices
add all faces so you have broken Δ^n into a bunch of n-simplicies

e.g.



exercise: Show "upto boundaries you can subdivide simplicies"

that is if $\sigma: \Delta^n \rightarrow X$ a singular n-simplex

and if $\Delta^n = \Delta_1^n \cup \dots \cup \Delta_k^n$ is its barycentric subdivision

then $\exists (n+1)$ -chain τ such that

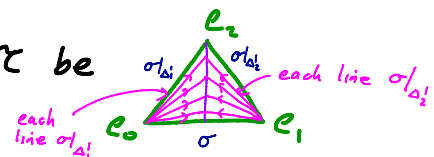
$$\sigma + \partial \tau = \sum_{i=1}^k \pm \sigma|_{\Delta_i^n}$$

↖ determine sign

Hint: for $\Delta^1 = \begin{array}{c} | \\ e_0 \end{array} \text{---} \begin{array}{c} | \\ e_1 \end{array}$

$$\Delta^1 = \Delta_1^1 \cup \Delta_2^1 \quad \begin{array}{c} | \\ f \\ \text{---} \\ e_0 \quad e_1 \end{array}$$

let τ be



$$\begin{aligned} \text{so } \partial\tau &= \tau|_{[e_0, e_1]} - \tau|_{[e_0, e_2]} + \tau|_{[e_0, e_1]} \\ &= \sigma - \sigma|_{\Delta_1} + \sigma|_{\Delta_2} \end{aligned}$$

note: as you repeatedly barycentrically subdivide a simplex the size of the resulting simplicies goes to zero.

now given a singular n -simplex $\sigma: \Delta^n \rightarrow X$

note $\{\text{int}A, X - \bar{z}\}$ is an open cover for X

$\therefore \{\sigma^{-1}(\text{int}A), \sigma^{-1}(X - \bar{z})\}$ is an open cover of Δ^n

$\therefore \exists$ a Lebesgue number $\delta > 0$ for the cover s.t. any set of diameter $< \delta$ is mapped by σ to $\text{int}A$ or $X - \bar{z}$

barycentrically subdivide Δ^n till each simplex has diam $< \delta$

for exercise $\exists \tau \in C_{n+1}(X)$ s.t. $\sigma + \partial\tau = \sum \pm \sigma|_{\text{subsimplices}}$

so we can replace σ with simplicies satisfying \circledast

\therefore we can do this for any $\sum m_i \sigma_i \in C_n(X, A)$ 